ON THE LEFSCHETZ NUMBER AND THE EULER CLASS

BY

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Let M be a compact connected orientable differentiable n-manifold (with empty boundary) and let $H^*(M)$ denote its singular cohomology group (integer coefficients). Thom [3] defined an element $X \in H^n(M)$ called the *Euler class* of M which has the property of being equal to the Euler characteristic of M times the fundamental class of M. Fadell [1] extended the definition of Euler class to compact connected orientable topological (i.e., not necessarily triangulated) manifolds. Let M be such an n-manifold, let $f: M \to M$ be a map, and let G be a commutative ring with unit. By modifying Milnor's development of the Euler class [2], we will define an endomorphism of $H^n(M; G)$ (singular cohomology with G coefficients) which maps the fundamental class of M to an element $L_f(G) \in H^n(M; G)$ which, when $G \cong Z$ (the integers), is equal to $(-1)^n$ times the Lefschetz number of f times the fundamental class of M. In particular, when f is the identity map, $L_f(Z)$ is Fadell's generalized Euler class. We shall obtain Thom's result in this more general setting. A simple proof of the Lefschetz Fixed-Point Theorem for this category of spaces also arises naturally from our development.

1. **Preliminaries.** Let (E, p, B) and (E_0, p_0, B) be Hurewicz fibre spaces over the same base space B. Fadell [1] defined (E_0, p_0, B) to be a *fibre subspace* of (E, p, B) provided $E_0 \subset E$, $p_0 = p \mid E_0$, and (E, p, B) admits a lifting function λ with the additional property that if $e_0 \in E_0$ and $w \in B^I$ such that $p(e_0) = w(0)$, then $\lambda(e_0, w) \in (E_0)^I$. When (E_0, p_0, B) is a fibre subspace of (E, p, B) Fadell calls $\mathfrak{F} = (E, E_0, p, B)$ a *fibred pair.* When $\mathfrak{F} = (E, E_0, p, B)$ is a fibred pair, the *fibre* over some $b \in B$ is the pair (F, F_0) where $F = p^{-1}(b)$ and $F_0 = F \cap E_0$.

A *generalized n-plane bundle* (n-gpb), $n \geqq 2$, is a map $p: E \to B$ together with a map $s: B \to E$ such that $ps = \mathrm{id}: B \to B$ and, if $E_0 = E - s(B)$, then

(1) (E, E_0, p, B) is a fibred pair with fibre (F, F_0),

(2) there is a homotopy $H: F \times I \to F$ such that $H(F \times [0,1)) \subset F_0$ and $H(F \times 1) = F \cap s(B)$,

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(3) F_0 is arcwise connected and, when $n \geqq 3$, $\pi_1(F_0) = 0$,

(4) $H_*(F, F_0) \cong H_*(E^n, E^n - 0)$, where E^n denotes Euclidean n-dimensional space and 0 is the origin of E^n.

This definition is slightly more general than that used by Fadell [1].

THEOREM 1.1 (FADELL). *Let $\mathfrak{F} = (E, E_0, p, B; F, F_0)$ be a fibred pair such that*

(1) *B is arcwise connected,*

(2) *$\pi_1(B, b)$ acts trivially on $H^*(F, F_0) = H^*(p^{-1}(b), p_0^{-1}(b))$,*

(3) *$H^*(F, F_0) \cong H^*(E^n, E^n - 0)$;*

then there exist natural "Thom" isomorphisms

$$\phi : H^q(B; H^n(F, F_0)) \to H^{n+q}(E, E_0),$$

where ordinary coefficients appear on the left. The inclusion map $i:(F, F_0) \to (E, E_0)$ induces an isomorphism $i^*: H^n(E, E_0) \to H^n(F, F_0)$. If, in addition, $\mathfrak{F}$ is an n-gpb, *then*

$$\begin{tikzcd} H^q(B; H^n(F, F_0)) = H^q(B) \arrow[r, "\phi"] \arrow[dr, "p^*"'] & H^{n+q}(E, E_0) \\ & H^q(E) \arrow[u, "\cup \mathfrak{U}"'] \end{tikzcd}$$

is commutative, where $i^*(\mathfrak{U})$ is a generator of $H^n(F, F_0) \cong Z$ (the integers).

Let M be a compact connected topological n-manifold. Define $p: M \times M \to M$ by $p(x, y) = x$ and let $\Delta = \{(x, y) \in M \times M \mid x = y\}$. Fadell proved in [1] that $\mathfrak{F} = (M \times M,\ M \times M - \Delta,\ p,\ M)$ is a locally trivial fibred pair with fibre $(p^{-1}(x), p_0^{-1}(x)) = (M, M - x)$ where $p_0 = p \mid M \times M - \Delta$. The lifting function of $\mathfrak{F}$ induces an action of $\pi_1(M; x)$ on $H^*(M, M - x)$. Fadell defined M to be *orientable* if this action is trivial and showed that in that case the inclusion-induced homomorphism $k^*: H^n(M, M - x) \to H^n(M)$ is an isomorphism. Let G be a commutative ring with unit. If M is orientable, it is G-orientable, so

$$k^*: H^n(M, M - x; G) \to H^n(M; G)$$

is an isomorphism. Choosing a generator $\bar{\mu} \in H^n(M, M - x; G)$, we obtain a generator $\mu \in H^n(M; G)$ by letting $\mu = k^*(\bar{\mu})$.

The fibred pair $\mathfrak{F} = (M \times M, M \times M - \Delta, p, M)$ satisfies the hypotheses of the first part of Theorem 1.1 when M is an orientable n-manifold. Therefore, the inclusion i: $(M, M - x) \to (M \times M, M \times M - \Delta)$ induces an isomorphism

$$i^*:\ H^n(M \times M, M \times M - \Delta; G) \to H^n(M, M - x; G).$$

Given a map f: $M \to M$ where M is a compact orientable n-manifold, let $j:(M \times M) \to (M \times M, M \times M - \Delta)$ be inclusion, let $\bar{f}$: $M \times M \to M \times M$ be given by $\bar{f}(y,z)=(y,f(z))$ and let $d\colon M \to M \times M$ be the diagonal map $d(y)=(y,y)$. Diagram (1) defines a homomorphism λ_G and we define the *Lefschetz class* of $f\colon M \to M$ (for cohomology with coefficient ring G), $L_f(G) \in H^n(M;G)$, by $\lambda_G(\mu)=L_f(G)$. We write $L_f(Z)=L_f$.

$$
(1)\qquad
\begin{array}{ccc}
H^n(M;G) & \xrightarrow{\quad\lambda_G\quad} & H^n(M;G) \\
k^*\big\uparrow\approx & & \big\uparrow d^* \\
H^n(M,M-x;G) & & H^n(M \times M;G) \\
i^*\big\uparrow\approx & & \big\uparrow \bar{f}^* \\
H^n(M\times M, M\times M-\Delta;G) & \xrightarrow[j^*]{} & H^n(M\times M;G)
\end{array}
$$

If $f\colon M \to M$ is fixed-point free then $\bar{f}d(M) \subset M \times M - \Delta$. Let $e\colon M \times M - \Delta \to M \times M$ be inclusion, then in this case $\bar{f}d = e\bar{f}d$ and $\lambda_G = d^*\bar{f}^*e^*j^*i^{*-1}k^{*-}$. From the exact cohomology sequence of the pair $(M \times M, M \times M - \Delta)$, we know that

$$e^*j^*=0\colon H^n(M\times M, M\times M-\Delta;G) \to H^n(M\times M-\Delta;G),$$

which proves:

THEOREM 1.2. *Let M be a compact orientable manifold and let G be a commutative ring with unit. If $f\colon M \to M$ is fixed-point free, then $L_f(G)=0$.*

2. **The main theorem.** Let Q denote the rationals. We will examine the relationship between the Lefschetz class $L_f(Q)$ of a map f and the Lefschetz number Λ_f. The reader may easily verify

LEMMA 2.1. *If a_{ij}, b_{ij} and c_{ij} are elements of a field, $i,j=1,\cdots,m$, then*

$$\sum_{i,j=1}^{m} a_{ij} \sum_{k=1}^{m} b_{ik}c_{jk} = \sum_{i,j=1}^{m} c_{ij} \sum_{k=1}^{m} b_{kj}a_{ki}.$$

The proof of the following theorem is based on a modification of the proof of Theorem 16 of [2].

THEOREM 2.2. *If M is a compact orientable n-manifold, $f\colon M \to M$ is a map, and $\mu \in H^n(M;Q)$ is the generator such that $\lambda_Q(\mu)=L_f(Q)$, then $L_f(Q) = (-1)^n\Lambda_f \cdot \mu$.*

Proof. Let $j^*i^{*-1}k^{*-1}(\mu) = U \in H^n(M \times M;\, Q)$ and extend μ to a basis $\alpha_1,\cdots,\alpha_N$ for $H^*(M,Q)$. If $\alpha_j \in H_q(M;Q)$ we say that the *dimension* of α_j is q, written $\dim(\alpha_j)=q$. By the Künneth formula we may write $U = \sum_{i,j=1}^{N} c_{ij}(\alpha_i \otimes \alpha_j)$ where $c_{ij}=0$ if $\dim(\alpha_i)+\dim(\alpha_j) \neq n$. Define $y_{ij} \in Q$ by $\alpha_i \cup \alpha_j = y_{ij}\cdot\mu$ if $\dim(\alpha_i)+\dim(\alpha_j)=n$ and $y_{ij}=0$ otherwise. Let f_{ij} be the ijth entry in the matrix representing f^*: $H^*(M;Q)\to H^*(M;Q)$ with respect to this basis. Note that $f_{ij}=0$ if $\dim(\alpha_i)\neq\dim(\alpha_j)$. Let $C=[c_{ij}]$, $Y=[y_{ij}]$ and $F=[f_{ij}]$ be the $N\times N$ matrices thus defined. Now

$$\begin{aligned} L_f(Q) &= d^*\bar{f}^*U = d^*\bar{f}^*\sum_{i,j=1}^{N} c_{ij}(\alpha_i\otimes\alpha_j) \\ &= \sum_{i,j=1}^{N} c_{ij}\sum_{k=1}^{N} f_{jk}(\alpha_i\cup\alpha_k) \\ &= \left(\sum_{i,j=1}^{N} c_{ij}\sum_{k=1}^{N} y_{ik}f_{jk}\right)\cdot\mu. \end{aligned}$$

Let $Y^T=[y_{ij}^T]$ be the transpose of Y, then applying 2.1,

$$\begin{aligned} L_f(Q) &= \left(\sum_{i,j=1}^{N} f_{ij}\sum_{k=1}^{N} y_{jk}^T c_{ki}\right)\cdot\mu \\ &= \left(\sum_{i,j=1}^{N} f_{ij}\,(y^Tc)_{ji}\right)\cdot\mu = \left(\sum_{i=1}^{N} a_{ii}\right)\cdot\mu, \end{aligned}$$

where $Y^TC=[(y^Tc)_{ij}]$ and $[a_{ij}]=A=FY^TC$. Renumber the basis elements of $H^*(M;Q)$ of dimension q as $\alpha_{q_1},\cdots,\alpha_{q_r}$, then by duality the basis elements of dimension $n-q$ are $\alpha_{(n-q)_1},\cdots,\alpha_{(n-q)_r}$. By the skew-symmetry of cup product, $y_{jk}^T=(-1)^e y_{jk}$, where $e=(\dim\alpha_j)(\dim\alpha_k)$, thus

$$\begin{aligned} \sum_{i=1}^{r} a_{q_i,q_i} &= \sum_{i=1}^{r}\sum_{j=1}^{N} f_{q_i,j}\sum_{k=1}^{N} y_{jk}^T c_{k,q_i} \\ &= \sum_{,j=1}^{r} f_{q_i,q_j}\sum_{k=1}^{r} y_{q_j,(n-q)_k}^T c_{(n-q)_k,q_i} \\ &= (-1)^{q(n-q)}\sum_{i,j=1}^{r} f_{q_i,q_j}\sum_{k=1}^{r} y_{q_j,(n-q)_k} c_{(n-q)_k,q_i}. \end{aligned}$$

Let $Y_q=[y_{q_i,(n-q)_j}]$, $C_q=[c_{(n-q)_i,q_j}]$, $i,j=1,\cdots,r$. By [2, p. 50], $C_qY_q=(-1)^{n(n-q)}E$, where E is the $r\times r$ identity matrix. Hence $Y_qC_q=(-1)^{n(n-q)}E$ and

$$\sum_{i=1}^{r} a_{q_i,q_i}=(-1)^{q(n-q)}\sum_{j=1}^{r}(-1)^{n(n-q)}f_{q_j,q_j}=(-1)^n\sum_{j=1}^{r}(-1)^q f_{q_j,q_j}.$$

Therefore

$$L_f(Q)=\left(\sum_{i=1}^{N} a_{ii}\right)\cdot\mu=\left((-1)^n\sum_{k=1}^{N}(-1)^{\dim(\alpha_k)}f_{kk}\right)\cdot\mu = (-1)^n\Lambda_f\cdot\mu.$$

3. The Lefschetz number.

THEOREM 3.1. *Let f: $M\to M$ be a map of a compact connected orientable n-manifold into itself; then the Lefschetz number of f is an integer.*

Proof. Let $i\colon Z \to Q$ be inclusion, then i induces a monomorphism $\bar{\imath}\colon H^n(M) \to H^n(M;\, Q)$. Let μ be a generator of $H^n(M)$, then the image of μ under $\bar{\imath}$, which we also call μ, is a generator of $H^n(M;\, Q)$. The restriction of λ_Q to $\bar{\imath}(H^n(M)) \subset H^n(M;\, Q)$ is just λ_Z so

$$L_f(Q) = \lambda_Q(\mu) = \bar{\imath}(\lambda_Z(\mu)) \in H^n(M),$$

which implies that $\Lambda_f \cdot \mu = (-1)^n L_f(Q)$ is in $\bar{\imath}(H^n(M))$ and $\Lambda_f \in Z$.

Putting together 2.2, 1.2, and 3.1 we obtain:

Theorem 3.2 (Lefschetz Fixed-Point Theorem). *If M is a compact connected orientable n-manifold and $f\colon M \to M$ is a map, then the Lefschetz number of f, Λ_f, is an integer such that if $\Lambda_f \neq 0$, then f has a fixed point.*

4. **The Euler class.** Let $\mathfrak{F} = (E, E_0, p, B; F, F_0)$ be an n-gpb where B is arcwise connected and $\pi_1(B; x)$ acts trivially on $H^*(F, F_0)$, then by Theorem 1.1 there exist isomorphisms $\phi\colon H^i(B) \to H^{n+i}(E, E_0)$ with $\phi(z) = p^*(z) \cup \mathfrak{U}$. In [1], the *Euler class* of $\mathfrak{F}$, $X(\mathfrak{F}) \in H^n(B)$, is defined by $X(\mathfrak{F}) = \phi^{-1}(\mathfrak{U} \cup \mathfrak{U})$. When M is a manifold, let

$$T_0 = \{\alpha \in M^I \mid \alpha(t) = \alpha(0) \text{ if and only if } t = 0\},$$

let T be the union of T_0 and the constant paths on M, and give T the compact-open topology. Define $q\colon T \to M$ by $q(\alpha) = \alpha(0)$, then by [1], $\mathfrak{F} = (T, T_0, q, M; F, F_0)$ is an n-gpb and the *Euler class* of M, $X(M) \in H^n(M)$, is defined by $X(M) = X(\mathfrak{F})$. The following result is based on results and techniques in [1] and [2]. We content ourselves with a sketch of the proof.

Theorem 4.1. *Let M be a compact connected orientable n-manifold, then for an appropriately chosen orientation of M, $X(M) = L_f$ where $f = \mathrm{id}\colon M \to M$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
(T, T_0) \arrow[rr, "\gamma"] \arrow[dr, "q"'] & & (M \times M, M \times M - \Delta), \arrow[dl, "p"] \\
& M &
\end{tikzcd}$$

where for $\alpha \in T$, $\gamma(\alpha) = (\alpha(0), \alpha(1))$. Then γ induces isomorphisms

$$\gamma^*\colon H^*(M \times M, M \times M - \Delta) \to H^*(T, T_0),$$

and when we restrict γ to the fibre we also have isomorphisms

$$\gamma^*\colon H^*(M, M - x) \to H^*(F, F_0).$$

Choose an orientation $\bar{\mu} \in H^n(M, M - x)$ and let $U = \gamma^*(\bar{\mu})$, then $i_1^*(\mathfrak{U}) = U$ where $i_1^*\colon H^n(T, T_0) \to H^n(F, F_0)$ is the isomorphism induced by inclusion. Let

$s: M \to T$ be the canonical cross-section, i.e., $s(y)$ is the constant path at y for all $y \in M$. Then, if $j_1: T \to (T, T_0)$ is the inclusion, $X(M) = s^* j_1^* (\mathfrak{U})$. Diagram (2) completes the proof.

$$
\begin{tikzcd}
H^n(F, F_0) & & H(M, M - x) \arrow[ll, "\gamma^*"', "\approx"] \\
H^n(T, T_0) \arrow[u, "i_1^*", "\approx"'] \arrow[d, "j_1^*"'] & & H^n(M \times M, M \times M - \Delta) \arrow[ll, "\approx", "\gamma^*"'] \arrow[u, "\approx", "i^*"'] \arrow[d, "j^*"] \\
H^n(T) \arrow[dr, "s^*"'] & & H^n(M \times M) \arrow[ll, "\gamma^*"'] \arrow[dl, "d"] \\
& H^n(M) &
\end{tikzcd}
\tag{2}
$$

THEOREM 4.2. *If M is a compact connected orientable n-manifold, then $X(M) = \chi(M) \cdot \mu$ where $\chi(M)$ is the Euler characteristic of M.*

Proof. By 2.2 and 3.1, $L_f = (-1)^n \Lambda_f \cdot \mu$ so since $\Lambda_{\mathrm{id}} = \chi(M)$, the result follows for n even. When n is odd, $\chi(M) = 0$.

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